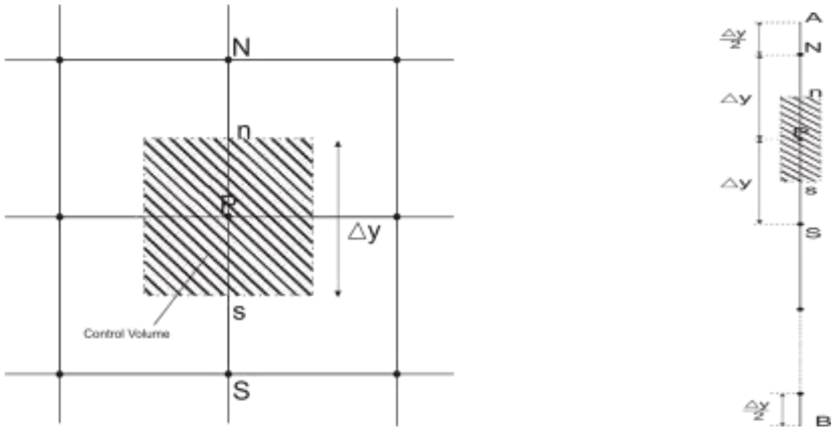
Supporting Information

Standard Solution Procedure for the Developed Model

The CVs and the nodes are defined on the basis of a numerical grid as shown in Fig. A1(a and b) where point P is the grid and points N and S are North and South neighbors respectively for the one-dimensional model being considered.



(a) (b)

Fig. A1: (a) Locations of the control-volume faces for the discretization. (b) Grid used in developing discretized

Applying the Finite Volume Difference, FVD method to non-dimensionalized Eq. [14].

Integration form of the equation with respect to non-dimension-time, T and the control volume, and the X is the non-dimension height of the medium.

$$\int_{T}^{T+\Delta T}\int_{n}^{s}\left[\frac{\partial W}{\partial T}\right]AdXdT = \int_{T}^{T+\Delta T}\int_{n}^{s}\left[D\frac{\partial^2 w}{\partial X^2}\right]AdXdT - \int_{T}^{T+\Delta T}\int_{n}^{s}\left[v\frac{\partial W}{\partial X}\right]AdXdT - \int_{T}^{T+\Delta T}\int_{n}^{s}\left[\beta^2\frac{W}{H^i+W}\right]AdXdT \qquad [A1]$$

Integrating over the control volume and dividing through by $A\Delta X$ and integrating over non-dimension time, T gives Eq. [A2]

$$\frac{\Delta X}{\Delta T}\left(W_p - W_p^o\right) = \theta\left[\left(\frac{W_S - W_p}{\delta X_{PS}}\right) - \left(\frac{W_p - W_N}{\delta X_{NP}}\right)\right] + (1-\theta)\left[\left(\frac{W_S^o - W_p^o}{\delta X_{PS}}\right) - \left(\frac{W_P^o - W_N^o}{\delta X_{NP}}\right)\right] - \left[\theta\left(Pe_s W_s - Pe_n W_n\right) + (1-\theta)\left(Pe_s^o W_s^o - Pe_n^o W_s\right)\right] - \beta^m\frac{W}{H^i+W}\Delta X \qquad [A2]$$

Applying the Fully Implicit method, which is recommended for general-purpose transient problems because of its robustness and unconditional stability, setting $\theta = 1$. The stability for any size of time is attributed to the fact that all coefficients are positive, and its accuracy is first order in time, small time steps are needed to ensure the accuracy of the results. This gives Eq. [A3]

$$\frac{\Delta X}{\Delta T}\left(W_p - W_p^o\right) = \left[\left(\frac{W_S - W_p}{\delta X_{PS}}\right) - \left(\frac{W_p - W_N}{\delta X_{NP}}\right)\right] - \left[\left(Pe_s W_s - Pe_n W_n\right) + \right] - \beta^m\frac{W}{H_a+W}\Delta X \qquad [A3]$$

Applying the Quick scheme for convection terms in the equation and considering only the positive direction of flow (due to gravity), that is $F_s > 0$, and $F_n > 0$.

$$\frac{\Delta X}{\Delta T}\left(W_P - W_P^o\right) = \left[\frac{1}{\delta X_{PS}}\left(W_S - W_P\right) - \frac{1}{\delta X_{NP}}\left(W_P - W_N\right)\right] - \left[\begin{array}{l}\left(\frac{6}{8}W_P + \frac{3}{8}W_S - \frac{1}{8}W_N\right)Pe_s \\ -\left(\frac{6}{8}W_N + \frac{3}{8}W_P - \frac{1}{8}W_{NN}\right)Pe_n\end{array}\right] \quad \text{[A4]}$$

$$-\beta^m \frac{W}{H^i + W}\Delta X$$

Let $\delta X_{PN} = \delta X_{SP} = \Delta X \;\; Pe_s = Pe_n = Pe$ and b is the linearized form of the source term and collect like the term

$$\left(\frac{\Delta X}{\Delta T} + \frac{1}{\Delta X} + \frac{1}{\Delta X} + \frac{6Pe}{8} - \frac{3Pe}{8}\right)W_P = \left(\frac{1}{\Delta X} - \frac{3Pe}{8}\right)W_S + \left(\frac{1}{\Delta X} + \frac{Pe}{8} + \frac{6Pe}{8}\right)W_N + \frac{\Delta X}{\Delta T}W_P^o - \frac{Pe}{8}W_{NN} - b \quad \text{[A5]}$$

Linearization of the source term using the standard format

$$\overline{S} = \overline{S}^* + \left(\frac{\partial S}{\partial w}\right)^* \left(W_P - W_P^o\right)$$

Where $S = \dfrac{\beta^m W}{H^i + W}$ [A6]

Differentiating Eq. [A6] with respect to W and substituting into the standard format give Eq. [A7] and [A8] respectively.

$$\frac{\partial S}{\partial W} = \frac{\left(H^i + W_P^o\right)\beta^m - \beta^m W_P^o .1}{\left(H^i + W_P^o\right)^2} = \frac{\beta^m H^i}{\left(H^i + W_P^o\right)^2} \quad \text{[A7]}$$

$$\overline{S} = \frac{\beta^m W_P^o}{H^i + W_P^o} + \frac{\beta^m H^i}{\left(H^i + W_P^o\right)^2}\left(W_P - W_P^o\right) \quad \text{[A8]}$$

Therefore,

$$S_c = \frac{\beta^m H^i}{H^i + W_P^o} + \left(\frac{\beta^m H^i}{\left(H^i + W_P^o\right)^2}\right)W_P \quad \text{[A9]}$$

$$S_P = \frac{\beta^m H^i}{\left(H^i + W_P^o\right)^2} \text{ and } S_u = \frac{\beta^m W_P^{0^2}}{\left(H^i + W_P^o\right)^2} \quad \text{[A10]}$$

By simplification of equation [A5], the Equation becomes,

$$\left(\frac{8\Delta X^2+16\Delta T+3Pe\Delta T\Delta X-S_P\Delta T\Delta X}{8\Delta T\Delta X}\right)W_P=\left(\frac{8-3Pe\Delta X}{8\Delta X}\right)W_S+\left(\frac{8+7Pe\Delta X}{8\Delta X}\right)W_N$$
$$+\frac{\Delta X}{\Delta T}W_P^o-\frac{Pe}{8}W_{NN}+b \tag{A11}$$

Where S_p is the Source term obtained from the linearization of the source term.

The general discretization equation in standard form for neighboring nodes (except for the first and last nodes which have to be treated separately) is

$$\alpha_P W_P=\alpha_S W_S+\alpha_N W_N+\alpha_P^o W_P^o+\alpha_{NN}W_{NN}+S_u \tag{A12}$$

Discretization at the First node, A

$$\frac{\Delta X}{\Delta T}\left(W_P-W_P^o\right)=\left[\frac{1}{\Delta X}\left(W_S-W_P\right)-\frac{1}{3\Delta X}\left(9W_P-8W_A-W_S\right)\right]$$
$$-\begin{bmatrix}\left(\frac{7}{8}W_P+\frac{3}{8}W_S-\frac{1}{8}W_A\right)Pe_s\\ -W_A Pe_a\end{bmatrix}+b \tag{A13}$$

By a collection of like terms, Eq. [A13] becomes

$$\left(\frac{\Delta X}{\Delta T}+\frac{1}{\Delta X}+\frac{3}{\Delta X}+\frac{7Pe}{8}\right)W_P=\left(\frac{1}{\Delta X}+\frac{1}{3\Delta X}-\frac{3Pe}{8}\right)W_S+\left(\frac{8}{3\Delta X}+\frac{Pe}{8}+Pe\right)W_A+\frac{\Delta X}{\Delta T}W_P^o$$
$$+b \tag{A14}$$

By simplifying Eq. [A14] results in Eq. [A15]

$$\left(\frac{8\Delta X^2+32\Delta T+7Pe\Delta T\Delta X-8S_p\Delta T\Delta X}{8\Delta T\Delta X}\right)W_p=\left(\frac{32-9Pe\Delta X}{24\Delta X}\right)W_S+\frac{\Delta X}{\Delta T}W_P^o+$$
$$\left(\frac{64+27Pe\Delta X}{24\Delta X}\right)W_A+b \tag{A15}$$

The discretization equation for the first node is given in Eq. [A16]

$$\alpha_P W_P=\alpha_S W_S+\alpha_A W_A+\alpha_P^o W_P^o+S_u \tag{A16}$$

Discretization at the Second Node,

The discretization equation at the second Node is given in Eq. [A17]

$$\frac{\Delta X}{\Delta T}\left(W_p-W_p^o\right)=\left[\frac{1}{\Delta X}\left(W_S-W_P\right)-\frac{1}{\Delta X}\left(W_P-W_N\right)\right]-$$
$$\begin{bmatrix}\left[\left(\frac{6}{8}W_p+\frac{3}{8}W_S-\frac{1}{8}W_A\right)Pe_s^+\right]\\ -\left[\left(\frac{7}{8}W_N+\frac{3}{8}W_P-\frac{2}{8}W_A\right)Pe_n^+\right]\end{bmatrix}-b \tag{A17}$$

By collection of like terms and simplification in Eq. [A17], the equation becomes

$$\left(\frac{8\Delta X^2+16\Delta T+3Pe^{+}\Delta T\Delta X-8S_P\Delta T\Delta X}{8\Delta T\Delta X}\right)W_P=\left(\frac{8-3Pe^{+}\Delta X}{24\Delta X}\right)W_S+\left(\frac{8+8Pe^{+}\Delta X}{8\Delta X}\right)W_N$$
$$+\left(\frac{\Delta X}{\Delta T}\right)W_p^o-\frac{Pe}{4}W_A+b \quad [A18]$$

Discretization at the Last node, B

The discretization equation at the last Node is shown in Eq. [A19]

$$\frac{\Delta X}{\Delta T}\left(W_p-W_p^o\right)=\left[\frac{1}{3\Delta X}\left(8W_B-9W_P+W_N\right)-\frac{1}{\Delta X}\left(W_P-W_N\right)\right]-$$
$$\left[\left[W_B Pe_b^{+}\right]-\left[\left(\frac{6}{8}W_N+\frac{3}{8}W_P-\frac{1}{8}W_{NN}\right)Pe_n^{+}\right]\right]-b \quad (A19)$$

Collection of like terms and by simplification the Equation becomes Eq. [A20]

$$\left(\frac{24\Delta X^2+32\Delta T-9Pe\Delta T\Delta X-24S_P\Delta T\Delta X}{24\Delta T\Delta X}\right)W_P=\left(\frac{8-3Pe\Delta X}{3\Delta X}\right)W_B+\left(\frac{32+18Pe\Delta X}{24\Delta X}\right)W_N+$$
$$\left(\frac{\Delta X}{\Delta T}\right)W_p^o-\left(\frac{1}{8}Pe\right)W_{NN}+b \quad [A20]$$

The equation of the last node.

$$\alpha_P W_P=\alpha_B W_B+\alpha_N W_N+\alpha_P^o W_P^o+\alpha_{NN}W_{NN}+S_u \quad (A21)$$

Implementation of the boundary conditions

The third-type boundary condition applies to the diffusion flux through boundary B equal to zero, and the value W at the boundary is equal to the upstream nodal value, that is $W_B=W_P$. The discretization equation becomes an equation shown in Eq. [A22].

$$\frac{\Delta X}{\Delta T}\left(W_p-W_p^o\right)=\left[0-\left(\frac{W_P}{\Delta X}-\frac{W_N}{\Delta X}\right)\right]-\left[\left[W_P Pe_b\right]-\left[\left(\frac{6}{8}W_N+\frac{3}{8}W_P-\frac{1}{8}W_{NN}\right)Pe_s\right]\right]-b \quad [A22]$$

$$\left(\frac{8\Delta X^2+8\Delta T+5Pe\Delta X\Delta T-8S_P\Delta X\Delta T}{8\Delta X\Delta T}\right)W_P=\left(\frac{4+3Pe}{4\Delta X}\right)W_N+\frac{\Delta X}{\Delta T}W_p^o-\frac{Pe}{8}W_{NN}-b \quad [A23]$$

The standard equation at the last node is shown in Eq. [A24]

$$\alpha_P W_P=\alpha_N W_N+\alpha_P^o W_P^o+\alpha_{NN}W_{NN}+S_u \quad [A24]$$

APPENDIX B: Flowchart for the Matlab Computation

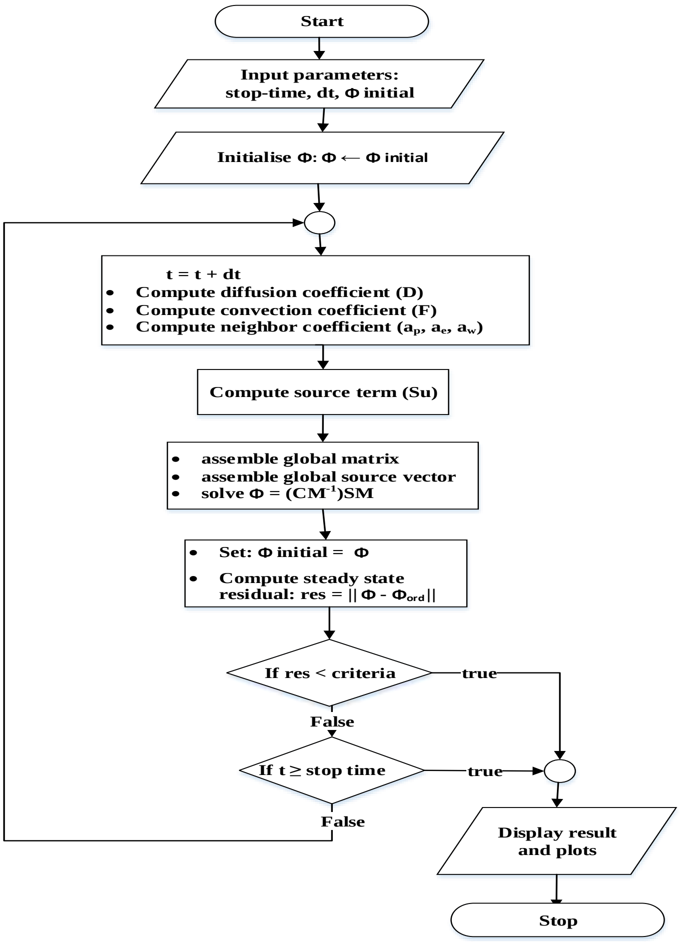


Fig. B1: Algorithm flow for the computer programme.

Note:

$$\left\|\boldsymbol{\Phi} - \boldsymbol{\Phi}_{ord}\right\| = \sqrt{\sum (\boldsymbol{\Phi}_i - \boldsymbol{\Phi}_{old(i)})^2}$$

$$CM = \begin{bmatrix} a_p^1 & -a_E^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_N^2 & a_p^2 & -a_E^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a_N^3 & a_p^3 & -a_E^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_N^4 & a_p^4 & -a_E^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_N^5 & a_p^5 & -a_E^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_N^6 & a_p^6 & -a_E^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_N^7 & a_p^7 & -a_E^7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_N^8 & a_p^8 & -a_E^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_N^9 & a_p^9 & -a_E^9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_N^{10} & a_p^{10} \end{bmatrix} \qquad SM = \begin{bmatrix} S_u^1 \\ S_u^2 \\ S_u^3 \\ S_u^4 \\ S_u^5 \\ S_u^6 \\ S_u^7 \\ S_u^8 \\ S_u^9 \\ S_u^{10} \end{bmatrix}$$