Appendix: A (Mathematical Model of Quadruple Tank Process)

Let F_1, F_2 be the flow rates of pump 1 and pump 2 respectively.

Let F_{ij} be the fraction of water flowing from Pump i to tank j.

Let F_{0j} be the outlet flowrate of tank j.

$$F_{11} = \gamma_1 F_1 \tag{1}$$

$$F_{14} = (1-\gamma_1)F_1 \qquad F_{22} = \gamma_2 F_2 \qquad F_{23} = (1-\gamma_2)F_2$$

Assume a square root relationship between the outlet flowrate and level of each tank. Let β_j represent the outlet

$$F_{o1} = \beta_1\sqrt{h_1} \qquad F_{o2} = \beta_2\sqrt{h_2} \tag{2}$$

$$F_{o3} = \beta_3\sqrt{h_3} \qquad F_{o4} = \beta_4\sqrt{h_4}$$

Mass balance on Tank 1:

$$f_1 = \frac{dh_1}{dt} = \frac{\gamma_1 F_1}{A_1} + \frac{\beta_3\sqrt{h_3}}{A_1} - \frac{\beta_1\sqrt{h_1}}{A_1} \tag{3}$$

Mass balance on Tank 2:

$$f_2 = \frac{dh_2}{dt} = \frac{\gamma_2 F_2}{A_2} + \frac{\beta_4\sqrt{h_4}}{A_2} - \frac{\beta_2\sqrt{h_2}}{A_2} \tag{4}$$

Mass balance on Tank 3:

$$f_3 = \frac{dh_3}{dt} = \frac{(1-\gamma_2)F_2}{A_3} - \frac{\beta_3\sqrt{h_3}}{A_3} \tag{5}$$

Mass balance on Tank 4:

$$f_4 = \frac{dh_4}{dt} = \frac{(1-\gamma_1)F_1}{A_4} - \frac{\beta_4\sqrt{h_4}}{A_4} \tag{6}$$

Steady state solution:

$$f_1 = f_2 = f_3 = f_4 = 0$$

$$h_{3s} = \left[\frac{(1-\gamma_2)F_{2s}}{\beta_3}\right]^2 \tag{7}$$

$$h_{4s} = \left[\frac{(1-\gamma_1)F_{1s}}{\beta_4}\right]^2 \tag{8}$$

$$h_{1s} = \left(\frac{\gamma_1 F_{1s} + (1-\gamma_2)F_{2s}}{\beta_1}\right)^2 \tag{9}$$

$$h_{2s} = \left(\frac{\gamma_2 F_{2s} + (1-\gamma_1)F_{1s}}{\beta_2}\right)^2 \tag{10}$$

Appendix: B (Transfer function matrix of Quadruple Tank Process)

Defining the State Space Model of the process as:

$$\dot{\mathbf{X}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx}$$

Where,

$$\dot{\mathbf{X}} = \begin{pmatrix} \frac{d(h_1 - h_{1s})}{dt} \\ \frac{d(h_2 - h_{2s})}{dt} \\ \frac{d(h_3 - h_{3s})}{dt} \\ \frac{d(h_4 - h_{4s})}{dt} \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} h_1 - h_{1s} \\ h_2 - h_{2s} \\ h_3 - h_{3s} \\ h_4 - h_{4s} \end{pmatrix}; \quad \mathbf{u} = \begin{pmatrix} F_1 - F_{1s} \\ F_2 - F_{2s} \end{pmatrix}$$

Elements of A and B Matrices are obtained by linearization:

$$a_{11} = \frac{\partial f_1}{\partial h_1} = \frac{-1}{2\sqrt{h_{1s}}}\left(\frac{\beta_1}{A_1}\right); \qquad a_{12} = \frac{\partial f_1}{\partial h_2} = 0$$

$$a_{13} = \frac{\partial f_1}{\partial h_3} = \frac{\beta_3}{2A_1\sqrt{h_{3s}}}; \qquad a_{14} = \frac{\partial f_1}{\partial h_4} = 0$$

$$a_{21} = \frac{\partial f_2}{\partial h_1} = 0; \qquad a_{22} = \frac{\partial f_2}{\partial h_2} = \frac{-1}{2A_2\sqrt{h_{2s}}}(\beta_2)$$

$$a_{23} = \frac{\partial f_2}{\partial h_3} = 0; \qquad a_{24} = \frac{\partial f_2}{\partial h_4} = \frac{\beta_4}{2A_2\sqrt{h_{4s}}}$$

$$a_{31} = \frac{\partial f_3}{\partial h_1} = 0; \qquad a_{32} = \frac{\partial f_3}{\partial h_2} = 0$$

$$a_{33} = \frac{\partial f_3}{\partial h_3} = \frac{-\beta_3}{2A_3\sqrt{h_{3s}}}; \qquad a_{34} = \frac{\partial f_3}{\partial h_4} = 0$$

$$a_{41} = \frac{\partial f_4}{\partial h_1} = 0; \qquad a_{42} = \frac{\partial f_4}{\partial h_2} = 0$$

$$a_{43} = \frac{\partial f_4}{\partial h_3} = 0; \qquad a_{44} = \frac{\partial f_4}{\partial h_4} = \frac{-\beta_4}{2A_4\sqrt{h_{4s}}}$$

$$b_{11}=\frac{\partial f_1}{\partial F_1}=\frac{\gamma_1}{A_1};\quad b_{12}=\frac{\partial f_1}{\partial F_2}=0;\quad b_{21}=\frac{\partial f_2}{\partial F_1}=0$$

$$b_{22}=\frac{\partial f_2}{\partial F_2}=\frac{\gamma_2}{A_2};\quad b_{31}=\frac{\partial f_3}{\partial F_1}=0;\quad b_{32}=\frac{\partial f_3}{\partial F_2}=\frac{(1-\gamma_2)}{A_3}$$

$$b_{41}=\frac{\partial f_4}{\partial F_1}=\frac{(1-\gamma_1)}{A_4};\quad b_{42}=\frac{\partial f_4}{\partial F_2}=0$$

Elements of C Matrix:

Since only h1 and h2 act as the two measured variables:

$$\mathbf{C}=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}$$

Final form of the state space model:

$$\dot{\mathbf{x}}=\begin{bmatrix}\frac{-1}{T_1}&0&\frac{A_3}{A_1T_3}&0\\0&\frac{-1}{T_2}&0&\frac{A_4}{A_2T_4}\\0&0&\frac{-1}{T_3}&0\\0&0&0&\frac{-1}{T_4}\end{bmatrix}\mathbf{x}+\begin{bmatrix}\frac{\gamma_1}{A_1}&0\\0&\frac{\gamma_2}{A_2}\\0&\frac{1-\gamma_2}{A_3}\\\frac{1-\gamma_1}{A_4}&0\end{bmatrix}\mathbf{u}$$

Where,

$$\frac{1}{T_i}=\frac{\beta_i}{2A_i\sqrt{h_{is}}}\tag{11}$$

Transfer function matrix of the process is determined as:

$$G(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B$$

$$G(s)=\begin{bmatrix}\frac{\gamma_1c_1}{1+sT_1}&\frac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)}\\\frac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)}&\frac{\gamma_2c_2}{1+sT_2}\end{bmatrix}$$

$$G(s)=\begin{bmatrix}\frac{k_{11}}{(1+sT_1)}&\frac{k_{12}}{(1+sT_1)(1+sT_3)}\\\frac{k_{21}}{(1+sT_2)(1+sT_4)}&\frac{k_{22}}{(1+sT_2)}\end{bmatrix}$$

Where,

$$c_i=\frac{T_i}{A_i}\tag{12}$$

$$k_{ii}=\gamma_ic_i$$
$$k_{ij}=(1-\gamma_i)c_j$$

Zeros of the system:

The zeros of transfer function matrix are found out by equating the determinant of the matrix to 0.

$$\det G(s)=$$

$$\frac{c_1c_2\gamma_1\gamma_2}{\prod_{i=1}^{4}(1+sT_i)}\left[(1+sT_3)(1+sT_4)-\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2}\right]$$

$$(1+sT_3)(1+sT_4)-\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2}=0$$

Let, $\eta=\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2}$

$$(1+sT_3)(1+sT_4)-\eta=0$$

For the case when, $\gamma_1+\gamma_2=1$ one zero is at origin and other is in the left half-plane.

Determination of Transmission Zeros of the process:

$$(1+sT_3)(1+sT_4)-\eta=0$$

$$T_3T_4s^2+(T_3+T_4)s+(1-\eta)=0$$

$$s=\frac{-(T_3+T_4)\pm\sqrt{(T_3+T_4)^2-4T_3T_4(1-\eta)}}{2T_3T_4}$$

Let, $A=T_3T_4$

$$B=T_3+T_4$$

$$C=1-\eta$$

$$As^2+Bs+C=0$$

Let the roots be $z_{1,2}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$

Determinant $D=\sqrt{B^2-4AC}$

$$z_1=\frac{-B+D}{2A}$$

$$z_2=\frac{-B-D}{2A}$$

The value of C can be either negative or positive or zero, depending on the value of η.

Case A: If C=0

D=B

$z_1 = 0$

$z_1 = \frac{-B}{A}$

$C = 1 - \eta = 0$

On solving $(1-\gamma_1)(1-\gamma_2) = \gamma_1\gamma_2$

$\gamma_1 + \gamma_2 = 1$

Case B: when C<0

$-4AC > 0$

$B^2 - 4AC > B^2$

$D > B$

$z_1 = +ve$

$z_2 = -ve$

So, $1 - \eta < 0$

$\gamma_1 + \gamma_2 < 1$

For the system to be in non-minimum phase: $0 < \gamma_1 + \gamma_2 < 1$

Case C: when $C > 0$

$z_1 = -ve, z_2 = -ve$

$\gamma_1 + \gamma_2 > 1$

Maximum value of $\gamma_1 + \gamma_2 = 2$ and min value is zero.

For both the zeros in left half-plane (one at $-1/T_3$ and other at $-1/T_4$) i.e. system in minimum phase:

$1 < \gamma_1 + \gamma_2 < 2$.