APPENDIX A

Solution methodology of governing equations

Before solving the third order differential Equation (9), a Change of the variable is applied as bellow:

$$\xi = f' \rightarrow \xi' = f'' \rightarrow \xi'' = f'''$$

This formulation is necessary to eliminate the need for third-order difference and to obtain a tri-diagonal system of linear algebraic equations at a further stage of the analysis.

The first step in solving the system of nonlinear ordinary differential equations (9) and (12) is to convert them into a system of quasi-linear differential equations.

$$\frac{\partial \xi^{(k+1)}}{\partial \tau} + M_0 \xi''^{(k+1)} + M_1 \xi'^{(k+1)} + M_2 \xi^{(k+1)} + M_3 f^{(k)} = M_4$$

$$\frac{\partial \theta^{(k+1)}}{\partial \tau} + L_0 \theta''^{(k+1)} + L_1 \theta'^{(k+1)} + L_2 \theta^{(k+1)} + L_3 f^{(k+1)} = L_4$$

Where

$$M_0 = -\frac{\eta}{Re}$$

$$M_1 = -\frac{1}{Re\,\frac{f^{(k)}}{Re}}$$

$$M_2 = 2\xi^{(k)}$$

$$M_3 = -\frac{\xi'^{(k)}}{Re}$$

$$M_4 = \xi^{2\,(k)} - \frac{f^{(k)}}{Re'^{(k)}}$$

$$L_0 = -\frac{\eta}{Re\,Pr}$$

$$L_1 = -\frac{1}{Re\,Pr}$$

$$L_2 = \frac{dT_w(\tau)/d\tau}{T_w(\tau) - T_\infty}$$

$$L_3 = -\theta'^{(k)}$$

$$L_4 = -f\theta'^{(k)}$$

where *k* and *k + 1* are the iteration indices.

In the numerical analysis, we replace the boundary conditions $\xi^{(k+1)}(\infty,\tau) = 1, \theta^{(k+1)}(\infty,\tau) = 0$ by $\xi^{(k+1)}(\eta_e,\tau) = 1, \theta^{(k+1)}(\eta_e,\tau) = 0$.

where η_e is a sufficiently large value of η.

Then, the problem has been written in a finite-difference form. The interval $1 \le \eta \le \eta_e$ has been divided into (*N*-1) equal intervals and denotes the values of the dependent variables at $\eta_i = 1 + (i-1)h$ with the subscript $i (= 1, 2, . . . ,N)$ where $h = (\eta_e - 1)/(N-1)$. Substituting, as usual, the expressions

$$\xi'' = \frac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{h^2},\ \xi' = \frac{\xi_{i+1} - \xi_{i-1}}{2h},\ \theta'' = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2},\ \theta' = \frac{\theta_{i+1} - \theta_{i-1}}{2h}$$

$$\frac{\partial \xi_i}{\partial \tau} = \frac{\xi_i - \xi_i^{old}}{\Delta\tau},\ \frac{\partial \theta_i}{\partial \tau} = \frac{\theta_i - \theta_i^{old}}{\Delta\tau}$$

In to equations and using the boundary conditions, at every time step we obtain a system of linear algebraic equations in a tridiagonal form:

$$A_{2,2}\xi_2^{(k+1)} + A_{2,3}\xi_3^{(k+1)} = C_2 - A_{2,1}$$

$$A_{i,i-1}\xi_{i-1}^{(k+1)} + A_{i,i}\xi_i^{(k+1)} + A_{i,i+1}\xi_{i+1}^{(k+1)} = C_i (i=3,4,\ldots,N-2)$$

$$A_{N-1,N-2}\xi_{N-2}^{(k+1)} + A_{N-1,N-1}\xi_{N-1}^{(k+1)} = C_{N-1} - A_{N-1,N}$$

$$A'_{2,2}\theta_2^{(k+1)} + A'_{2,3}\theta_3^{(k+1)} = C'_2 - A'_{2,1}$$

$$A'_{i,i-1}\theta_{i-1}^{(k+1)} + A'_{i,i}\theta_i^{(k+1)} + A'_{i,i+1}\theta_{i+1}^{(k+1)} = C'_i (i=3,4,\ldots,N-2)$$

$$A'_{N-1,N-2}\theta_{N-2}^{(k+1)} + A'_{N-1,N-1}\theta_{N-1}^{(k+1)} = C'_{N-1} - A'_{N-1,N}$$

Where

$$A_{i,i-1} = \frac{M_0}{h^2} - \frac{M_1}{2h}$$

$$A_{i,i} = \frac{1}{\Delta\tau} - \frac{2M_0}{h^2} + M_2$$

$$A_{i,i+1} = \frac{M_0}{h^2} + \frac{M_1}{2h}$$

$$C_i = M_4 + \frac{\xi_i^{(old)}}{\Delta\tau} - M_3 f_i^{(k)}$$

$$A'_{i,i-1} = \frac{L_0}{h^2} - \frac{L_1}{2h}$$

$$A'_{i,i} = \frac{1}{\Delta\tau} - \frac{2L_0}{h^2} + L_2$$

$$A'_{i,i+1} = \frac{L_0}{h^2} + \frac{L_1}{2h}$$

$$C'_i = L_4 + \frac{\theta_i^{(old)}}{\Delta\tau} - L_3 f_i$$

In the above relations, the superscript (*old*) represents the calculated value of ξ and θ in the previous time step.

These systems are composed of (*N-2*)equations for (*N-2*) unknowns $\xi_i^{(k+1)}$, $\theta_i^{(k+1)}$. It can be solved quite easily by usual sweeping method. Once all of $\xi_i^{(k+1)}$ are determined, $f_i^{(k+1)}$ are obtained from $\xi = f'$, namely:

$$f_i^{(k+1)} = \int_1^{\eta_i} \xi_i^{(k+1)} d\eta$$

executing a numerical integration. The values of $\xi_i^{(k+1)}$ and $f_i^{(k+1)}$ obtained here are used to replace $\xi_i^{(k)}$ and $f_i^{(k)}$ for the next cycle. The convergence is considered achieved if $\left|\xi_i^{(k+1)} - \xi_i^{(k)}\right| \leq \varepsilon$ and $\left|\theta_i^{(k+1)} - \theta_i^{(k)}\right| \leq \varepsilon$ for all points, where ε is a prescribed accuracy criterion